Eur. Phys. J. B **22**, 361–368 (2001)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Nonlinear diffusion under a time dependent external force: q-maximum entropy solutions

C. Giordano[1], A.R. Plastino[1,2,3], M. Casas[3], and A. Plastino[2,4,a]

[1] Faculty of Astronomy and Geophysics, National University La Plata, CC 727, 1900 La Plata, Argentina
[2] Argentine National Research Council (CONICET), CC 727, 1900 La Plata, Argentina
[3] Departament de Física, Universitat de les Illes Balears, 07071 Palma de Mallorca, Spain
[4] Department of Physics, National University La Plata, CC 727, 1900 La Plata, Argentina

Received 25 April 2001 and Received in final form 6 June 2001

Abstract. Nonlinear diffusion equations provide useful models for a number of interesting phenomena, such as diffusion processes in porous media. We study here a family of nonlinear Fokker-Planck equations endowed both with a power-law nonlinear diffusion term and a drift term with a time dependent force linear in the spatial variable. We show that these partial differential equations exhibit exact time dependent particular solutions of the Tsallis maximum entropy (q-MaxEnt) form. These results constitute generalizations of previous ones recently discussed in the literature [C. Tsallis, D.J. Bukman, Phys. Rev. E **54**, R2197 (1996)], concerning q-MaxEnt solutions to nonlinear Fokker-Planck equations with linear, time independent drift forces. We also show that the present formalism can be used to generate approximate q-MaxEnt solutions for nonlinear Fokker-Planck equations with time independent drift forces characterized by a general spatial dependence.

PACS. 66.10.Cb Diffusion and thermal diffusion – 05.20.-y Classical statistical mechanics – 05.60.-k Transport processes – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

Research conducted during the last few years on the thermostatistical aspects of anomalous nonlinear diffusion processes [1–13] revealed the existence of interesting links between the phenomenon of nonlinear diffusion, on the one hand, and Tsallis nonextensive thermostatistical formalism [14,15] on the other one. In particular, it was discovered that Tsallis maximum entropy distributions provide useful ansaetze for arriving at both exact and approximate solutions of various nonlinear partial differential equations describing processes involving anomalous diffusion [1,2,8,9]. These Tsallis MaxEnt distributions are obtained *via* the extremization, with appropriate constraints, of Tsallis nonextensive entropy,

$$S_q = \frac{1}{q-1}\left(1 - \int \rho(\mathbf{x})^q \, \mathrm{d}\mathbf{x}\right), \qquad (1)$$

where $\mathbf{x} \in R^N$ is a dimensionless state-variable, $\rho(\mathbf{x})$ stands for the probability distribution describing the system, and the Tsallis parameter q is any real number. The standard Boltzmann-Gibbs logarithmic entropy $S = -\int \rho(\mathbf{x}) \ln \rho(\mathbf{x}) \mathrm{d}\mathbf{x}$ is recovered in the limit $q \to 1$. The measure S_q is *nonextensive*. That is, the entropy of a composite system $A \oplus B$ constituted by two subsystems A and B, which are statistically independent in the sense that $\rho(\mathbf{x}, \mathbf{x}')_{A\oplus B} = \rho(\mathbf{x})_A \, \rho(\mathbf{x}')_B$, is not equal to the sum of the individual entropies associated with each subsystem. Instead, the entropy of the composite system is given by Tsallis' q-additive relation,

$$S_q(A \oplus B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B). \qquad (2)$$

The above equation implies that the value of Tsallis' parameter q determines the degree of nonextensivity exhibited by S_q. Many of the physically relevant mathematical properties of the standard thermostatistics are either verified by Tsallis' nonextensive formalism, or can be appropriately generalized [14,15]. In particular, Tsallis' proposal was shown to be consistent with Jaynes' information theory re-formulation of statistical mechanics [16].

The first clue suggesting that the q-MaxEnt principle may be useful for obtaining exact or approximate solutions of certain important non linear partial differential equations was given in [8], where the connection between Tsallis' formalism and nonlinear diffusion processes was established. It was there shown that the maximization

[a] e-mail: plastino@venus.fisica.unlp.edu.ar

of Tsallis' entropy subject to appropriate (simple) constraints provides exact time dependent solutions for a family of nonlinear Fokker-Planck equations characterized by a diffusion term depending on a power of the probability density. These nonlinear Fokker-Planck equations have been applied in the modelling of several physical phenomena, such as the percolation of gases through porous media [17], thin liquid films spreading under gravity [18], surface growth [19], and some processes of self-organized criticality [20]. In particular, they have been used to describe systems showing anomalous diffusion of the correlated type [6]. Reaction-diffusion equations with nonlinear diffusion have also attracted the attention of researchers [21,22] because they yield useful mathematical models for a number of interesting phenomena, like diffusion and recombination processes in plasma physics [23], and the kinetics of phase transitions [24].

In [8] we discussed analytical q-MaxEnt solutions of a one-dimensional nonlinear Fokker-Planck equation with a constant diffusion coefficient and a linear homogeneous drift force. These exact solutions maximize Tsallis entropy under the constraints imposed by normalization and the mean values of x and x^2, and are of the form

$$\rho(x,t) \,=\, N\left[1 \,-\, \beta\,(1-q)\;(x-x_0)^2\right]^{1/(1-q)}, \qquad (3)$$

where N, β, and x_0 are time dependent parameters (see Sect. 3 for a detailed discussion). Solutions of the form (3) constitute natural generalizations (in the sense of Tsallis' q-formalism) of the celebrated Gaussian solutions of a (linear) Ornstein-Uhlenbeck process [25]. Of course, these are particular solutions of the (nonlinear) partial differential equation under consideration. That is, they constitute only a subset of all the possible solutions. However, they are important because they are *exact analytical solutions*. Nonlinear partial differential equations are very difficult to solve either analytically or numerically. The knowledge of exact (particular) analytical solutions is always valuable, because they may provide some insight on what to expect with regards to the behavior of more general solutions. On the other hand, to know exact analytical solutions is very helpful in order to check the accuracy of numerical procedures developed to solve the concomitant nonlinear evolution equation.

A detailed study of this kind of solutions, within a more general scenario, was given by Tsallis and Bukman in [1]. Tsallis entropy turned out to be unique, in the sense of being the only non logarithmic measure providing Gaussian-like, MaxEnt time dependent solutions for an associated family of non linear Fokker-Planck equations [12]. A q-MaxEnt scheme generating approximate solutions of nonlinear Fokker-Planck equations with state dependent diffusion was developed in [9]. A microscopic (phenomenological) foundation for the nonlinear Fokker-Planck equation based upon a non extensive generalization of the Ito-Langevin dynamics was advanced by Borland [4]. An interesting discussion of the nonextensive thermostatistical aspects of nonlinear diffusion in connection with the second law of thermodynamics has been recently provided by Frank [5]. Drazer, Wio and Tsallis (DWT) applied the nonextensive q-formalism to a nonlinear Fokker-Planck equation with an absorption term, and obtained exact solutions of the q-MaxEnt form [2]. The absorption term in the DWT equation acts as a (negative) source term that leads to a non-conservation of the norm of the concomitant solutions. Nonlinear reaction-diffusion equations with a different kind of source (reaction) term were also shown to be endowed with Tsallis MaxEnt exact solutions [10]. The equations studied in [10] have a nonlinear diffusion term together with a (also nonlinear) logistic-like reaction term. A more general evolution equation with nonlinear diffusion, linear drift, and a Verhulst-like term was considered in [11] in connection with Tsallis formalism.

The aim of the present effort is twofold. On the one hand, we are going to study particular, exact time dependent q-MaxEnt solutions for a nonlinear Fokker-Planck equation with a time dependent drift force linear in the spatial variable. On the other hand, we are going to show that these q-MaxEnt solutions can be used to obtain approximate q-MaxEnt solutions to nonlinear Fokker-Planck equations with a time independent drift force exhibiting a general spatial dependence.

The paper is organized as follows: first of all, a brief description of the nonlinear Fokker-Planck equation is provided in Section 2. In Section 3 we study q-MaxEnt exact solutions for the nonlinear Fokker-Planck equation with time dependent drift. Related approximate solutions for the nonlinear Fokker-Planck equation with time independent drift forces characterized by a general spatial dependence are considered in Section 4. Finally, some conclusions are drawn in Section 5.

2 The nonlinear Fokker-Planck equation with a time dependent potential

We are going to study nonlinear Fokker-Planck equations of the form,

$$\frac{\partial \rho}{\partial t} \,=\, D\,\frac{\partial^2}{\partial x^2}\,\rho^{\delta} \,-\, \frac{\partial}{\partial x}\left[\rho\left(-\frac{\partial V}{\partial x}\right)\right], \qquad (4)$$

where $\rho(x,t)$ stands for an appropriately normalized probability distribution. The evolution of ρ is governed by two terms. On the one hand we have the diffusion term

$$D\,\frac{\partial^2\left(\rho^{\delta}\right)}{\partial x^2}, \qquad (5)$$

which describes the effect of stochastic forces characterized by the diffusion coefficient D. The quantities D and δ are usually required to be positive. However, the less restrictive condition $\delta\, D > 0$ still leads to a physically meaningful evolution. On the other hand we have the drift term

$$\frac{\partial}{\partial x}\left[\rho\left(-\frac{\partial V}{\partial x}\right)\right], \qquad (6)$$

due to the deterministic (drift) force

$$K(x) = -\frac{\partial V}{\partial x} \tag{7}$$

arising from the potential function $V(x,t)$. We are going to consider a time dependent potential function quadratic in the spatial variable x,

$$V(x,t) = \alpha(t)\,\frac{x^2}{2} + \gamma(t)\,x, \tag{8}$$

where the coefficients α and γ are known functions of time.

3 Tsallis MaxEnt solutions

The maximum entropy principle provides a powerful method for obtaining reduced descriptions of the dynamics of evolving systems. This approach is based on the study of the behavior of a small number of relevant mean values and adopts, for describing the system under consideration, the probability distribution function that maximizes the entropy under the constraints imposed by normalization and the mean values of some relevant quantities. Such MaxEnt ideas, within the standard Shannon-Jaynes framework, have been applied to a variety of evolution equations (see for instance [26,27] and references therein). As already mentioned in the Introduction, this MaxEnt approach to time dependent problems has also been investigated with regards to more general situations, where the system under consideration is appropriately described by probability distribution functions maximizing Tsallis' generalized entropy (see [9] for a detailed account). In particular, the q-MaxEnt prescription has provided exact (particular) solutions for diverse evolution equations associated with nonlinear diffusion processes.

Here, we are going to obtain exact time dependent solutions of the Tsallis MaxEnt form for the nonlinear Fokker-Planck equation (4) with the time dependent potential (8). Following [1,8,9] we are going to consider the ansatz

$$\rho(x,t) = N\left[1 - \beta\,(1-q)\,(x-x_0)^2\right]^{1/(1-q)}, \tag{9}$$

where q is Tsallis' parameter and N, β, and x_0 are time dependent parameters. As we shall presently see (and in accordance with previous work on q-MaxEnt solutions to nonlinear diffusion equations [8]) the Tsallis' parameter q leading to exact time dependent solutions of the nonlinear Fokker-Planck equation (4) is related to the exponent δ by

$$q = 2 - \delta. \tag{10}$$

This means that the appropriate value of Tsallis' parameter q is determined by the evolution equation (4) itself. This constitutes an interesting situation: the value of Tsallis' parameter is clearly determined *by the dynamics of the problem under consideration.* Notice that in the limit $q \to 1$ the q-MaxEnt distribution (9) adopts the standard Gaussian shape

$$\rho(x,t) = N\exp[-\beta\,(x-x_0)^2]. \tag{11}$$

This limit case is associated with standard linear diffusion, which corresponds to $\delta = 1$.

Let us replace the ansatz (9) in the nonlinear Fokker-Planck equation (4) (with the potential $V(x)$ given by (8)). Equating now in both members of (4) the coefficients associated with corresponding powers of x, it is possible to show that the q-MaxEnt distribution (9) is an exact solution of the evolution equation (4) provided that the parameters x_0, N, and β, satisfy the system of coupled ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x_0(t) = -\,\alpha(t)\,x_0(t) - \gamma(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,N(t) = \alpha(t)\,N(t) - 2D(2-q)\beta(t)\,N^{(2-q)}(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\beta(t) = 2\alpha(t)\,\beta(t) - 4D(2-q)\;\beta^2(t)\,N^{(1-q)}(t). \tag{12}$$

The equation of motion for x_0 does not depend on the Tsallis parameter q. This equation can be readily solved by quadratures. Its solution is

$$x_0(t) = \mu^{-1}(t)\left[\mu(t_0)\,x_0(t_0) - \int_{t_0}^{t}\mu(\tau)\,\gamma(\tau)\;\mathrm{d}\tau\right], \tag{13}$$

where

$$\mu(t) = \exp\left(\int \alpha(t)\,\mathrm{d}t\right). \tag{14}$$

The system of equations (12) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) = \frac{2\beta}{N}\,\frac{\mathrm{d}}{\mathrm{d}t}N(t), \tag{15}$$

which leads to

$$N(t) = N(t_0)\;\left[\frac{\beta(t)}{\beta(t_0)}\right]^{1/2}. \tag{16}$$

By recourse to the above equation and to the third equation of the system (12) we obtain the following equation of motion for the variable β,

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) = 2\,\alpha(t)\,\beta(t) - 2\,B\,\beta^{(5-q)/2}(t), \tag{17}$$

where the quantity B does not depend on time and is given, in terms of the initial conditions, by

$$B = 2\,D\,(2-q)\left(\frac{N(t_0)}{\sqrt{\beta(t_0)}}\right)^{(1-q)}. \tag{18}$$

In order to obtain the solution of (17) it is necessary to consider separately the cases $q = 3$ and $q \neq 3$.

3.1 The case q = 3

In this case equation (17) acquires the simpler form

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) = 2\left[\alpha(t) + \frac{2D\beta(t_0)}{N^2(t_0)}\right]\beta(t), \tag{19}$$

which can be integrated giving

$$\beta(t) = \beta(t_0)\,\frac{\mu^2(t)}{\mu^2(t_0)}\,\exp\left[\frac{4D\beta(t_0)}{N^2(t_0)}(t-t_0)\right]. \tag{20}$$

3.2 The case q ≠ 3

When $q \neq 3$ it is convenient to introduce the new variable

$$Y = \beta^{(q-3)/2}, \tag{21}$$

which verifies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + (3-q)\,\alpha(t)\,Y(t) = (3-q)\,B. \tag{22}$$

The above equation admits the solution

$$Y(t) = \mu^{(q-3)}(t) \times\left[\mu^{(3-q)}(t_0)\,Y(t_0) + (3-q)\,B\int_{t_0}^{t}\mu^{(3-q)}(\tau)\,\mathrm{d}\tau\right], \tag{23}$$

leading to

$$\beta(t) = \mu^2(t)\left[\mu^{(3-q)}(t_0)\,\beta^{(q-3)/2}(t_0) + (3-q)\,B\int_{t_0}^{t}\mu^{(3-q)}(\tau)\mathrm{d}\tau\right]^{2/(q-3)}. \tag{24}$$

Finally, from the solutions (20) or (24) for $\beta(t)$, and the equation (16) relating N to β, it is possible to obtain $N(t)$.

3.3 Example: Time dependent periodic drift force

As an illustration of the Tsallis MaxEnt solutions to the nonlinear Fokker-Planck equation with a time dependent potential, let us consider the potential

$$V(x,t) = \alpha_1\left(1+\frac{\cos\omega t}{2}\right)\frac{x^2}{2} + \alpha_2\,x. \tag{25}$$

In what follows we are going to assume that $q \neq 3$. From equations (14) and (25) it follows that

$$\mu(t) = \exp\left(\alpha_1 t + \frac{\alpha_1}{2\omega}\sin\omega t\right). \tag{26}$$

It is possible, after some algebra, to verify that the time dependence of x_0 (see Eq. (13)) associated with the potential (25) is

$$\begin{aligned} x_0(t) = \exp\left(-\frac{\alpha_1}{2\omega}\sin\omega t\right)\Big\{ & x_0(0)\exp(-\alpha_1 t) \\ & + \alpha_2\Big[a_1\Big(\exp(-\alpha_1 t) - 1\Big) \\ & + a_2\Big(\omega(\cos\omega t - \exp(-\alpha_1 t)) - \alpha_1\sin\omega t\Big) \\ & + a_3\Big(\alpha_1(\cos 2\omega t - \exp(-\alpha_1 t)) \\ & + 2\omega\sin 2\omega t\Big)\Big]\Big\}, \end{aligned} \tag{27}$$

where

$$a_1 = \frac{1}{\alpha_1}\left[1+\left(\frac{\alpha_1}{4\,\omega}\right)^2\right]$$

$$a_2 = \frac{\alpha_1}{2\,\omega\,(\alpha_1^2+\omega^2)},$$

$$a_3 = \frac{\alpha_1^2}{8\,\omega^2\,(\alpha_1^2+4\omega^2)}\,. \tag{28}$$

The differential equation (22) admits the closed analytical solution

$$\begin{aligned} Y(t) = \exp\left(-\frac{\alpha_1}{2\omega}\sin\omega t\right)\Big\{ & Y(0)\exp((3-q)\alpha_1 t) \\ & + (3-q)B\Big[b_1\Big(1-\exp((3-q)\alpha_1 t)\Big) \\ & + b_2\Big(\omega(\exp((3-q)\alpha_1 t) - \cos\omega t) \\ & - (3-q)\alpha_1\sin\omega t\Big) \\ & + b_3\Big((3-q)\alpha_1(\cos 2\omega t - \exp((3-q)\alpha_1 t)) \\ & - 2\omega\sin 2\omega t\Big)\Big]\Big\}, \end{aligned} \tag{29}$$

with

$$b_1 = \frac{1}{(3-q)\alpha_1}\left[1+\frac{(3-q)^2\,\alpha_1^2}{16\,\omega^2}\right]$$

$$b_2 = \frac{(3-q)\,\alpha_1}{2\,\omega\,[(3-q)^2\alpha_1^2+\omega^2]},$$

$$b_3 = \frac{(3-q)^2\,\alpha_1^2}{8\,\omega^2\,[(3-q)^2\alpha_1^2+4\omega^2]}, \tag{30}$$

leading, in turn, to the following closed expression for β,

$$\beta(t) = \exp\left[\frac{\alpha_1}{(3-q)\omega}\sin\omega t\right]\left\{\beta^{(q-3)/2}(0)\exp\left(\frac{2\alpha_1 t}{(3-q)}\right) + (3-q)B\Big[b_1\Big(1-\exp\left((3-q)\alpha_1 t\right)\Big)\right.$$
$$\left. + b_2\Big((q-3)\alpha_1\sin\omega t + \omega(\cos\omega t - \exp\left((3-q)\alpha_1 t\right))\Big) + b_3\Big(2\omega\sin 2\omega t + (q-3)\alpha_1(\exp\left((3-q)\alpha_1 t\right) - \cos 2\omega)\Big)\Big]\right\}^{2/(q-3)} \cdot \quad (31)$$

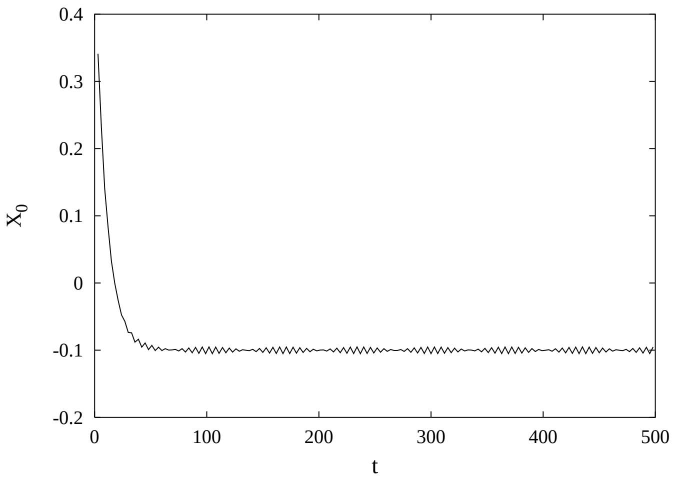

Fig. 1. The time evolution of x_0 corresponding to the solution of the nonlinear Fokker-Planck equation for $q = 0.5$ and diffusion coefficient $D = 0.05$, when the linear drift is due to the time-dependent potential $V(x) = 0.1[1 + 0.5\cos(2t)]x^2 + 0.1\,x$.

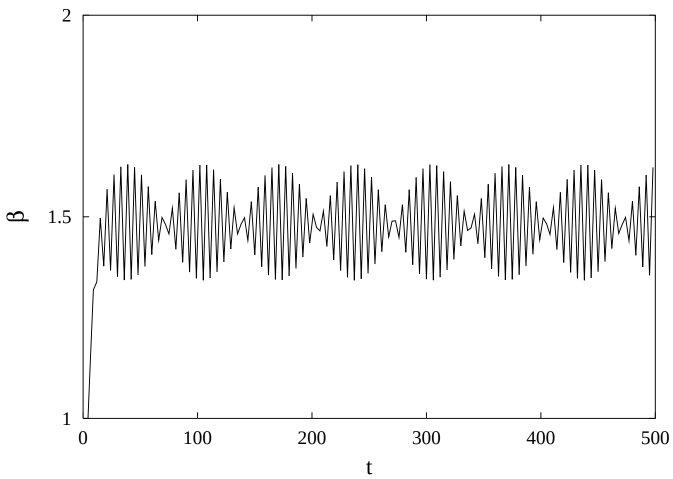

Fig. 2. The time evolution of β corresponding to the solution of the nonlinear Fokker-Planck equation for $q = 0.5$ and diffusion coefficient $D = 0.05$, when the linear drift is due to the time-dependent potential $V(x) = 0.1[1 + 0.5\cos(2t)]x^2 + 0.1\,x$.

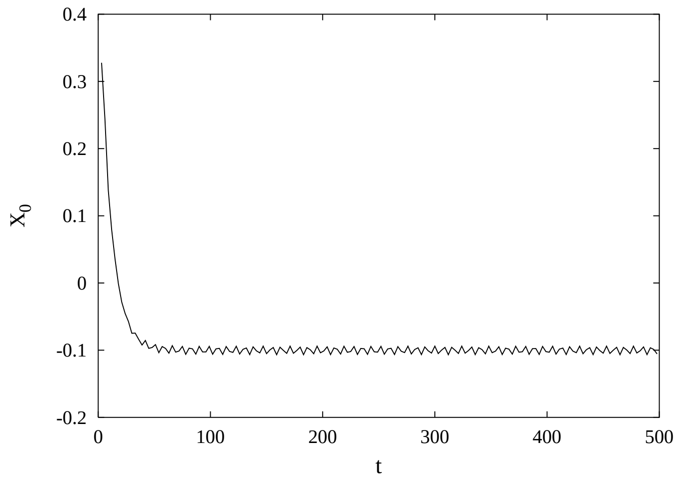

Fig. 3. The time evolution of x_0 corresponding to the solution of the nonlinear Fokker-Planck equation for $q = 0.5$ and diffusion coefficient $D = 0.05$, when the linear drift is due to the time-dependent potential $V(x) = 0.1[1 + 0.5\cos(2.34t/3)]x^2 + 0.1\,x$.

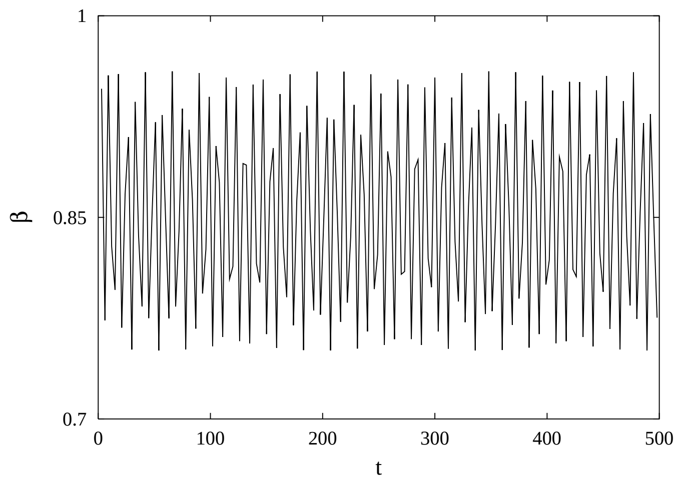

Fig. 4. The time evolution of β corresponding to the solution of the nonlinear Fokker-Planck equation for $q = 0.5$ and diffusion coefficient $D = 0.05$, when the linear drift is due to the time-dependent potential $V(x) = 0.1[1 + 0.5\cos(2.34t/3)]x^2 + 0.1\,x$.

See equation (31) above.

Equations (27) and (31), together with the relation (16), completely determine the time evolution of the q-MaxEnt solution (9) of the nonlinear Fokker-Planck equation with the periodically time dependent potential (25).

Typical examples of the behavior of the parameters x_0 and β characterizing the Tsallis solutions associated with periodically time dependent potentials are depicted in Figures 1–4. Figures 1 and 2 correspond to $w = 2.0$, and Figures 3 and 4 to $w = 0.78$. As was shown in [1,8,9], when one has time independent potentials the solutions of the nonlinear Fokker-Planck equation evolve irreversibly towards an equilibrium q-MaxEnt stationary solution. It is clear that there exist no stationary solutions when time dependent potentials are present. In these cases, and after a transient, the solutions relax towards a periodically time dependent distribution $\rho(x,t)$ exhibiting the Tsallis' q-MaxEnt form. These asymptotic periodic distributions play, for periodic drift forces, a role similar to the one played by the Tsallis' q-MaxEnt stationary solutions associated with time independent potentials.

4 Localized, approximate, MaxEnt solutions for time independent potentials with a general space dependence

In this section we are going to show how the exact Tsallis MaxEnt solutions corresponding to time dependent drift forces can be used to obtain approximate q-MaxEnt solutions for nonlinear Fokker-Planck equations of the form (4), endowed with a drift force arising from a constant (that is, not depending on time) potential $V(x)$. This potential can be of quite general shape (dependence on the space variable x). Our approximation will describe *localized* solutions. That is, solutions $\rho(x,t)$ with a relatively small "width" in x. Let us consider again the Tsallis ansatz

$$\rho = N\left[1-\beta(1-q)(x-x_0)^2\right]^{1/(1-q)}, \qquad (32)$$

with $\rho = 0$ whenever the bracket above becomes negative (Tsallis cut-off condition [15]). Clearly, the distribution ρ is centered at x_0 and its width is determined by the value of the parameter β. The main assumption of our approximation will be that the interval where the distribution ρ differs appreciably from zero is small enough for the potential $V(x)$ to be adequately represented, in that region, by the second order Taylor expansion

$$V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{V''(x_0)}{2}(x-x_0)^2. \qquad (33)$$

Within these conditions, the nonlinear Fokker-Planck equation can be recast under the guise

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho^{(2-q)}}{\partial x^2} + \frac{\partial}{\partial x}\Big[\rho\left(V'(x_0)+V''(x_0)(x-x_0)\right)\Big]. \qquad (34)$$

Notice that the potential function appearing in the above equation *is time dependent*, for it depends on x_0 which, in general, is a function of time. However, in this case the time dependence of the (approximate) potential is not given beforehand. It is determined, in a self-consistent way, by the center x_0 of the evolving distribution $\rho(x,t)$. The present approach is akin to the celebrated "wave packet approximation" in quantum mechanics [28]. Replacing now the ansatz (32) in equation (34) it can be shown, after some algebra, that the former constitutes a solution of the latter provided that the parameters x_0, N, and β comply with the system of coupled differential equations

$$\begin{aligned}
\frac{\mathrm{d}x_0}{\mathrm{d}t} &= -V'(x_0)\\
\frac{\mathrm{d}N}{\mathrm{d}t} &= V''(x_0)N - 2D(2-q)\beta N^{(2-q)}\\
\frac{\mathrm{d}\beta}{\mathrm{d}t} &= 2V''(x_0)\beta - 4D(2-q)\beta^2\, N^{(1-q)}.
\end{aligned} \qquad (35)$$

By a procedure similar to the one discussed in the previous section it is possible to show that β evolves according to the differential equation

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = 2\,V''(x_0)\beta - 2\,B\,\beta^{(5-q)/2}, \qquad (36)$$

where B is a constant given by equation (18). As it was the case for Fokker-Planck equations with a time dependent drift, it is necessary now to consider separately the case $q = 3$. For that value of the Tsallis parameter the equation for β reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) = 2\left[V''(x_0) + \frac{2D\beta(t_0)}{N^2(t_0)}\right]\beta(t). \qquad (37)$$

If $q \neq 3$, it proves convenient to rewrite the equation for β in the fashion

$$\frac{\mathrm{d}Y}{\mathrm{d}t} + (3-q)V''(x_0)Y = (3-q)B, \qquad (38)$$

in terms of the quantity Y defined by the change of variables

$$Y = \beta^{(q-3)/2}; \quad q \neq 3. \qquad (39)$$

It follows from equations (35) (following the same steps discussed in Sect. 3) that the parameter N can be obtained from β by recourse to equation (16). As a result, the evolution of ρ is completely determined by equation (38) and by the first equation of the system (35), that governs the evolution of x_0. This pair of coupled ordinary differential equations does not have, in general, a closed analytical solution. As a consequence, a numerical treatment becomes necessary. Nevertheless, instead of dealing with a partial differential equation we face now the much easier task of solving a system of two ordinary differential equations. Furthermore, the functional dependence of the approximate solution (32) on the spatial variable x is of a completely analytical nature.

4.1 A numerical example

In order to illustrate the above scheme we are going to implement it numerically for the quartic potential

$$V(x) = x^2 + x^4. \qquad (40)$$

The behavior of the entropy's rate of growth provides a criterion for assessing the accuracy of MaxEnt-related approximations (see [9] and references therein). The time derivative of the Tsallis q-entropy (1) corresponding to an evolving probability distribution $\rho(x,t)$ is given by

$$\frac{\mathrm{d}S_q}{\mathrm{d}t} = \frac{q}{(1-q)} \int \rho^{(q-1)} \ \frac{\partial \rho}{\partial t} \ \mathrm{d}x. \tag{41}$$

Let $\rho_{\mathrm{T}}(x,t)$ stand for the approximate distribution given by the Tsallis ansatz (32) at a given instant t. The time derivative of the nonextensive entropy S_q at time t is then given by

$$\left(\frac{\mathrm{d}S_q}{\mathrm{d}t}\right)_{\mathrm{T}} = \frac{q}{(1-q)} \int \mathrm{d}x \, \rho_{\mathrm{T}}^{(q-1)} \times \left\{ D \frac{\partial^2 {\rho_{\mathrm{T}}}^{(2-q)}}{\partial x^2} + \frac{\partial}{\partial x} \Big[\rho_{\mathrm{T}} \left(V'(x_0) + V''(x_0)(x-x_0) \right) \Big] \right\}. \tag{42}$$

Now, suppose that we take the instantaneous distribution ρ at time t (that is $\rho_{\mathrm{T}}(x,t)$) and from that instant on we let ρ evolve according to its exact equation of motion (that is, the one involving the exact potential $V(x)$, instead of its truncated Taylor expansion (33)). Then, the time derivative of S_q at time t, evaluated under the exact evolution, would be

$$\left(\frac{\mathrm{d}S_q}{\mathrm{d}t}\right)_{\mathrm{TE}} = \frac{q}{(1-q)} \int \mathrm{d}x \, \rho_{\mathrm{T}}^{(q-1)} \times \left\{ D \, \frac{\partial^2 {\rho_{\mathrm{T}}}^{(2-q)}}{\partial x^2} + \frac{\partial}{\partial x} \left[\rho_{\mathrm{T}} \frac{\mathrm{d}V}{\mathrm{d}x} \right] \right\}. \tag{43}$$

An entropy's growth criterion for evaluating the quality of the MaxEnt approximation is obtained by recourse to the comparison of the two functions (of time) $\left(\frac{\mathrm{d}S_q}{\mathrm{d}t}\right)_{\mathrm{T}}(t)$ and $\left(\frac{\mathrm{d}S_q}{\mathrm{d}t}\right)_{\mathrm{TE}}(t)$. A specific numerical example is shown in Figure 5, where the functions $\left(\frac{\mathrm{d}S_q}{\mathrm{d}t}\right)_{\mathrm{T}}(t)$ (dotted line) and $\left(\frac{\mathrm{d}S_q}{\mathrm{d}t}\right)_{\mathrm{TE}}(t)$ (solid line) are depicted. This figure also exhibits the time derivative of S_q associated with the truncated potential $V = x^2$ (dash-dotted line). It is clear from the figure that the local approximation provides a better description than the one obtained by just assuming that we are dealing with a quadratic potential.

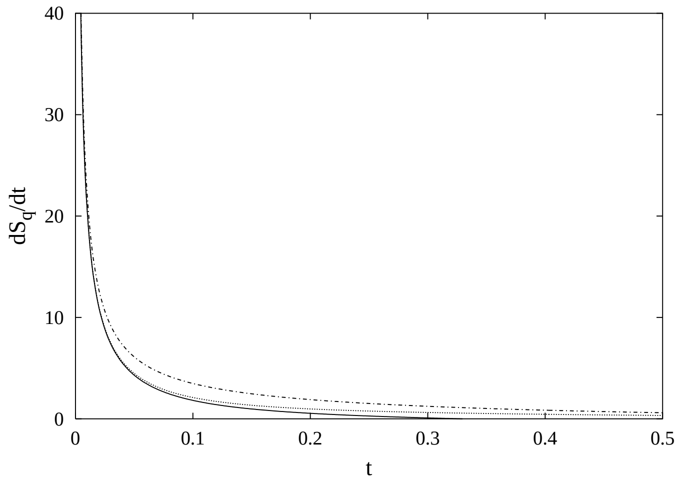

Fig. 5. The time-derivative of the Tsallis' generalized entropy for the case of a time-independent drift given by the potential energy $V(x) = x^2 + x^4$, for $q = 0.5$ and $D = 0.1$ (solid line). The dotted curve corresponds to the second order approximation for $V(x)$ in powers of $(x - x_0)$, while the dash-dotted curve has been obtained for the truncated potential $V(x) = x^2$. The initial values are $x_0(t_0) = 2$ and $\beta(t_0) = 3000$.

5 Conclusions

We studied particular exact solutions exhibiting the Tsallis' MaxEnt form of a nonlinear Fokker-Planck evolution equation characterized by a nonlinear power-like diffusion term and a drift term arising from a quadratic potential with time dependent coefficients. We showed that the dynamics of these q-MaxEnt solutions is governed by a system of three coupled nonlinear, ordinary differential equations (that is, Eqs. (12)), that can be solved by quadratures. As an illustration of these results we provided a detailed discussion of the particular case associated with a periodic potential.

We also applied the above developments to a scheme for obtaining approximate, localized, q-MaxEnt solutions for nonlinear Fokker-Planck equations with drift forces not depending on time, but characterized by a general spatial dependence. The evolution of the concomitant q-MaxEnt solutions is determined by three coupled ordinary differential equations that do not admit, in general, exact analytical solutions. We studied numerically the behavior of this system in a particular example.

The results reported here provide new evidence on the usefulness of the generalized statistical formalism (based on Tsallis' nonextensive entropic measure) as a tool for the study of nonlinear, power-law diffusion processes.

The present effort is based upon work partially supported by the AECI Scientific Cooperation Program, by the DGES grants PB98-0124 and SB97-26373862 (Spain), and by CONICET (Argentine Agency).

References

1. C. Tsallis, D.J. Bukman, Phys. Rev. E **54**, R2197 (1996).
2. G. Drazer, H.S. Wio, C. Tsallis, Phys. Rev. E **61**, 1417 (2000).
3. G. Kaniadakis, G. Lapenta, Phys. Rev. E **62**, 3246 (2000).
4. L. Borland, Phys. Rev. E **57**, 6634 (1998).

5. T.D. Frank, Phys. Lett. A **267**, 298 (2000).
6. A. Compte, D. Jou, J. Phys. A **29**, 4321 (1996).
7. A. Compte, D. Jou, Y. Katayama, J. Phys. A **30**, 1023 (1997).
8. A.R. Plastino, A. Plastino, Physica A **222**, 347 (1995).
9. L. Borland, F. Pennini, A.R. Plastino, A. Plastino, Eur. Phys. J. B **12**, 285 (1999).
10. A.R. Plastino, M. Casas, A. Plastino, Physica A **280**, 289 (2000).
11. A. Rigo, A.R. Plastino, M. Casas, A. Plastino, Phys. Lett. A **276**, 97 (2000).
12. S. Martinez, A.R. Plastino, A. Plastino, Physica A **259**, 183 (1998).
13. T.D. Frank, A. Daffertshofer, Physica A **272**, 497 (1999).
14. C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
15. C. Tsallis, Physics World **10**, 42 (1997); C. Tsallis, Braz. J. Phys. **29**, 1 (1999).
16. A.R. Plastino, A. Plastino, Phys. Lett. A **177**, 177 (1993).
17. M. Muskat, *The Flow of Homogeneous Fluids Through Porous Media* (McGraw-Hill, New York, 1937).
18. J. Buckmaster, J. Fluid Mech. **81**, 735 (1977).
19. H. Spohn, J. Phys. I France **3**, 69 (1993).
20. J.M. Carlson, E.R. Grannan, C. Singh, G.H. Swindle, Phys. Rev. E **48**, 688 (1993).
21. J.J.E. Herrera, A. Minzoni, R. Ondarza, Physica D **57**, 249 (1992).
22. R.D. Benguria, M.C. Depassier, Phys. Rev. E **52**, 3285 (1995).
23. A.C. Scott, Rev. Mod. Phys. **47**, 487 (1975).
24. H. Wilhelmsson, Phys. Rev. A **36**, 965 (1987); H. Wilhelmsson, Phys. Rev. A **38**, 1482 (1988); H. Wilhelmsson, Phys. Rev. A **38**, 2667 (1988).
25. H. Haken, *Synergetics* (Springer-Verlag, Berlin, 1983).
26. A.R. Plastino, H.G. Miller, A. Plastino, Phys. Rev. E **56**, 3927 (1997).
27. L. Borland, Physica D **99**, 175 (1996).
28. M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).